connection, an increase in the value of the parameter ε used in the calculations was limited to the maximum value of $\varepsilon_0^* = \varepsilon_l - 0.0005$. The value of H_0 determined for it can be regarded as an approximation to the value H_0^{**}.

Fig.4 shows the lines of separation calculated for the combinations of initial parameters used in the table. In the series with the value $\rho = 0.01$, the extreme versions are shown by dashed lines, and in the case of $\rho = 0.001$, by the dot-dash lines. The curve for $H_0 = 35, \rho = 0.001$ was not drawn, since it is very close to the line of separation at $H_0 = 35$, $\rho = 0.01$, $L = 50$. This is confirmed by comparing the versions in question with the tabulated values of H_1 and H_2. The destabilization of saline waters of reduced density occurs when the rate of evaporation is reduced almost proportionally, and hence also the process of filtration within the border zone. In conformity with this, the maximum depth T of the free surface of the border zone is also reduced. The depression curves formally resemble each other in all the computed versions, and their positions within the ranges of the graph also differ little from each other. For this reason only one such curve is included in Fig.4 for the lens at $L = 50, \rho = 0.01$.

The relationship of ε^* described above is illustrated in Fig.5 for $\rho = 0.01$ by the upper line $(L = 25)$ and lower line $(L = 50)$. The dashed horizontal lines show the corresponding limit values of the parameter $\varepsilon = \varepsilon_l$.

The authors thank N.S. Kolodei for help in carrying out the computations.

REFERENCES

1. BERESLAVSKII and EMIKH V.N., The problem of filtration from a system of channels at the border zone separating fresh waters from saline waters below, with evaporation. PMM, 47, 3, 1983.
2. POLUBARINOVA-KOCHINA P.YA., Theory of the Motion of Ground Waters. Moscow, Nauka, 1977.
3. JAHNKE E., EMDE F. and LOSH F., Tafeln höheren Funktionen. Stuttgart, Teubner, 1966.

Translated by L.K.

PMM U.S.S.R.,Vol.52,No.5,pp.683-685,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

ISOPERIMETRIC INEQUALITY IN THE PROBLEM OF THE STABILITY OF A CIRCULAR RING UNDER NORMAL PRESSURE*

V.V. KOBELEV

The problem of maximizing the critical load causing a loss of stability in an elastic inextensible circular ring under hydrostatic pressure is studied. An undeformed ring has the form of a circle of unit radius, and its thickness, and hence the flexural rigidity, varies along the arc. The thickness distribution must be determined from the condition of maximum critical load causing the loss of stability, under the condition that the mass of the ring remains constant. It is shown that of all circular rings of the same mass a ring of constant thickness can bear the greatest load before losing stability.

1. Basic equations and formulation of the problem of optimization. Let us consider the conditions for the loss of stability of a circular ring acted upon by a uniformly distributed, compressive hydrostatic load. We know that under the action of hydrostatic pressure the elementary load vectors remain normal to the curved axis of the ring, and the work done by this load is equal to the product of the pressure and the difference in the areas bounded by the ring in its deformed and undeformed state. Therefore, the external load is conservative, and the phenomenon of loss of stability can be studied using static methods.

**Prikl.Matem.Mekhan.*,52,5,871-873,1988

We shall assume that one of the principal central axes of symmetry of transverse cross-section lies in the plane of curvature of the ring. When the compresssive load reaches its critical value Λ, the initial circular form of the ring becomes unstable and a perturbed state of equilibrium appears. We shall assume that the bending occurs in the plane of curvature of the ring. Then the dimensionless critical load will be equal to the minimum value of the Rayleigh ratio /1/

$$\Lambda = \min R[\tau^n], \quad R[\tau^n] = \langle \tau^n (w'' + w)^2 \rangle / \langle w'^2 - w^2 \rangle \tag{1.1}$$

$$\langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(s)\, ds$$

The minimum in (1.1) is found from the set of all 2π-periodic, twice differentiable functions w such, that

$$\langle w \rangle = \langle w \sin s \rangle = \langle w \cos s \rangle = 0. \tag{1.2}$$

We note that w has the meaning of dimensionless normal deflection. We denote by $\tau = \tau(s)$ a dimensionless, 2π-periodic function proportional to the area of transverse cross-section of the ring, such that

$$\langle \tau \rangle = 1, \ \tau(s) > 0 \tag{1.3}$$

The first condition of (1.3) expresses the requirement that the mass of the ring must be constant. The quantity τ^n is proportional to the flexural rigidity of the ring, and the exponent n takes the values of 1, 2 and 3. The cases $n = 1$ and $n = 3$ describe the situations in which the form of transverse cross-section undergoes an affine transformation such that one of the gometrical dimensions of the cross-section (the width or height of the cross-section respectively) changes, while $n = 2$ corresponds to a congruent change in the form of the cross-section /3/.

The problem of optimization is formulated in the following manner. We require to find the distribution $\tau(s)$ satisfying the condition (1.3) such, that the critical load causing the loss of stability has its maximum value

$$\Lambda^* = \max_\tau \Lambda \tag{1.4}$$

We shall show that a ring of constant thickness is the optimal one, and that the following inequality holds:

$$\Lambda \leqslant \Lambda^* = 3 \tag{1.5}$$

where the equality is attained only when the ring is of constant thickness. The inequality (1.5) refers to isoperimetric inequalities related to the eigenvalues of the boundary value problems for ordinary differential equations. Some of the general methods of investigating isoperimetric inequalities are described in /3-5/.

2. Transforming the variational formulation, and general properties of the boundary value problem. Before proving inequality (1.5), we shall mention some facts which are essential for further discussion. We note that Λ is the smallest positive eigenvalue of the selfconjugate boundary value problem with periodic boundary conditions

$$y'' + y + \lambda\tau^{-n} y = 0 \qquad y(0) = y(2\pi), \quad y'(0) = y'(2\pi) \tag{2.1}$$

We can confirm this by writing out the Euler equations for the functional (1.1) and putting $y = \tau^n (w'' + w)$ in them.

We know (/6/, Ch.VI, Sect.3) that the eigenvalues of the boundary value problem (2.1) are simple or double, and that they can be arranged to form a non-decreasing sequence $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_i, \lambda_{i+1}, \ldots$ $(\lambda_i \leqslant \lambda_{i+1})$ so that the eigenfunction corresponding to the eigenvalue λ_i has, in the semi-interval $[0, 2\pi)$ i zeros if i is even, and $i + 1$ zeros if i is odd. Further, $\lambda_0 < \lambda_1 = \lambda_2 = 0$. The eigenfunctions $\cos s$ and $\sin s$ correspond to doubly degenerate eigenvalue equal to zero. The third and fourth eigenvalue have a physical meaning, namely, the dimensionless critical load causing loss of stability Λ is equal to the third eigenvalue $\Lambda = \lambda_3$, which may be double $\Lambda = \lambda_3 = \lambda_4$.

Let us denote by $u(s)$ and $v(s)$ the eigenfunctions of the boundary value problem (2.1) corresponding to the eigenvalues λ_3 and λ_4. These functions change their sign in the semi-interval $[0, 2\pi)$, four times. We shall assume that they are orthonormed

$$\langle u^2 \tau^{-n} \rangle = \langle v^2 \tau^{-n} \rangle = 1, \quad \langle uv\tau^{-n} \rangle = 0 \tag{2.2}$$

The eigenvalues of the boundary value problem (2.1) can be regarded as extremal values of the following variational problem:

$$\lambda_3 = \min_{y \in \Pi(\tau)} I[y] \equiv I[u] \qquad \lambda_4 = \min_{y \in \Pi_\perp(\tau)} I[y] \equiv I[v] \tag{2.3}$$

$$I[z] = 2\pi \langle z'^2 - z^2 \rangle$$

$$\Pi(\tau) = \{y \mid y \in C^2[0, 2\pi), \ \ y(s) = y(s + 2\pi)$$

$$\langle y \rangle = \langle y \sin s \rangle = \langle y \cos s \rangle = 0, \ \ \langle \tau^{-n} y^2 \rangle = 1\}$$

$$\Pi_\perp(\tau) = \{y \mid y \in \Pi(\tau), \ \ \langle yu\tau^{-n} \rangle = 0\}$$

In particular, we have

$$\Lambda_* = \min_{y\in\Pi(1)} I[y] = \min_{y\in\Pi_\perp(1)} I[y] = 2\pi\langle u_0'^2 - u_0^2\rangle = 2\pi\langle v_0'^2 - v_0^2\rangle = 3$$

$$u_0 = \sqrt{2}\cos 2s, \quad v_0 = \sqrt{2}\sin 2s$$

3. Proof of the isoperimetric inequality. We shall show that the inequality $\lambda_3 \leqslant \Lambda_*$ holds for any function $\tau(s)$ such that conditions (1.3) hold and the equality is attained only for $\tau(s) = 1$.

Indeed, since $\lambda_3 \leqslant \lambda_4$, we have the following relation for any congruence functions $u_1 \in \Pi(\tau)$, $u_2 \in \Pi(\tau)$:

$$\lambda_3 \equiv \frac{\lambda_3+\lambda_3}{2} \leqslant \frac{2\pi\langle u_1'^2 + u_2'^2 - u_1^2 - u_2^2\rangle}{\langle \tau^{-n}(u_1^2+u_2^2)\rangle} \tag{3.1}$$

In particular, we can put

$$\begin{aligned} &u_1 = C^{-1/2}(u_0\cos\theta + v_0\sin\theta) \\ &u_2 = C^{-1/2}(-u_0\sin\theta + v_0\cos\theta) \\ &2C = \langle\tau^{-n}(u_0^2+v_0^2)\rangle > 0 \\ &\cos 2\theta\,\langle\tau^{-n}(u_0^2-v_0^2)\rangle + \sin 2\theta\,\langle\tau^{-n}u_0v_0\rangle = 0 \end{aligned} \tag{3.2}$$

Then the right-hand side of inequality (3.1) will be equal to

$$\frac{2\pi\langle u_0'^2 + v_0'^2 - u_0^2 - v_0^2\rangle}{\langle\tau^{-n}(u_0^2+v_0^2)\rangle} \leqslant \frac{2\pi\langle u_0'^2 + v_0'^2 - u_0^2 - v_0^2\rangle}{\langle u_0^2+v_0^2\rangle} \equiv \Lambda_* \tag{3.3}$$

The latter inequality follows from (1.3). Indeed, since $u_0^2 + v_0^2 = 2$, therefore $\langle\tau^{-h}(u_0^2 + v_0^2)\rangle = 2\langle\tau^{-n}\rangle$. From the Holder inequality about the mean values (/7/, Theorem 211 where we put $r = -n$, $s = 1$) , we have

$$\langle\tau^{-n}\rangle^{-1/n} \leqslant \langle\tau\rangle = 1$$

From this it follows that $\langle\tau^{-n}\rangle \geqslant 1$ and the equality is attained only for the functions identically equal to unity. Thus the sequence of inequalities (3.1), (3.3) is established and the inequality (1.5) is thus proved.

Note. A general method of establishing the sufficient conditions for the local optimum and simple eigenvalues is developed in /8/. It is easy to confirm that the sufficient conditions for the local optimum are satisfied in the present problem.

Inequality (1.5) follows, in the case of $n = 1$, from a more general inequality. Inequality (1.5) remains valid when τ^{-n} in the Rayleigh functional is replaced by an arbitrary concave function $\Phi(\tau)$ for which $\Phi(\bar\tau) \geqslant \overline{\Phi(\tau)}$, $\Phi(1) = 1$ where $\bar\tau$ denotes the mean value (/7/, Ch.6). The linear function is obviously concave: $\Phi(\bar\tau) = \overline{\Phi(\tau)}$. Indeed, the proof follows from the sequence of inequalities

$$\Lambda_* = \min R[\Phi(\bar\tau)] \geqslant R[\overline{\Phi(\tau)}] = \overline{R[\Phi(\tau)]} \geqslant \Lambda$$

The first inequality follows from the definition of concave functions, and the second inequality from the extremal property of the eigenvalues. The minimum is chosen from amongst all admissible functions.

REFERENCES

1. BUDIANSKY B., FRAUENTHAL J.C. and HUTCHINSON J.W., An optimal arches, Trans. ASME, Ser. E. J. Appl. Mech. 36, 4, 1969.
2. BANICHUK N.V., Introduction to the Optimization of Structures. Nauka, Moscow, 1986.
3. BANDLE C., Isoperimetric inequalities and applications. Boston, Pitman, 1980.
4. PAYNE L.E., Isoperimetric inequalities and their applications, SIAM Rev. 9, 3, 1967.
5. TADJBAKHSH I. and KELLER J.B., Strongest columns and isoperimetric inequalities for eigenvalues, Trans. ASME, Ser. E. J. Appl. Mech., 29, 1, 1962.
6. SANSONE G., Ordinary Differential Equations. 1, IIL, Moscow, 1953.
7. HARDY G.H., LITTLEWOOD J.E. and POLYA G., Inequalities. Cambridge Univ. Press, 1964.
8. BRATUS A.S. and SEIRANYAN A.P., Sufficient conditions for an extremum in eigenvalue optimization problems. PMM, 48, 4, 1984.

Translated by L.K.